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Microscopic equation for growing interfaces in quenched disordered media

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Abstract. We present a microscopic equation for a growing interface with quenched noise of the Tang and Leschhorn model (Tang L H and Leschhorn H 1992 *Phys. Rev.* A **45** R8309). Evolution equations for the height, the mean height, and the roughness are reached in a simple way. An equation for the interface activity density (or free sites density) as a function of time is obtained. The microscopic equation allows us to express these equations in terms of two contributions: the diffusion and the substratum contributions. All these equations shows the strong interplay between the diffusion and the substratum contribution in the dynamics.

1. Introduction

The investigation of rough surfaces and interfaces has attracted much attention, for decades, due to its importance in many fields, such as the motion of liquids in porous media, growth of bacterial colonies, crystal growth, etc. Much effort has been devoted to understanding the properties in these processes [1]. When a fluid wets a porous medium, a nonequilibrium self-affine rough interface is generated. The interface has been characterized through scaling of the interfacial width $w = \langle [h_i - \langle h_i \rangle]^2 \rangle^{1/2}$ with time t and lateral size L. The result is the determination of two exponents, β and α, called dynamical and roughness exponents, respectively. The interfacial width $w \sim L^{\alpha}$ for $t \gg L^{\alpha/\beta}$, and $w \sim t^{\beta}$ for $t \ll L^{\alpha/\beta}$. The crossover time between these two regimes is of the order of $L^{\alpha/\beta}$.

The formation of interfaces is determinated by several factors; it is very difficult to theoretically discriminate all of them. An understanding of the dynamical nonlinearities, the disorder of the media, and the theoretical model representing experimental results is difficult to arrive at due the complex nature of the growth. The disorder affects the motion of the interface and leads to its roughness. Two main kinds of disorder have been proposed: the 'annealed' noise that depends only on time and the 'quenched' disorder due to the inhomogeneity of the media in which the moving phase is propagating. Some experiments, such as the growth of bacterial colonies and the motion of liquids in porous media, where the disorder is quenched, are well described by the directed percolation depinning model. This model was proposed simultaneously by Tang and Leschhorn (TL) [2] and Buldyrev *et al* [3]. Braunstein and Buceta [4] showed that the power-law scaling for the roughness only holds at criticality for $t \ll L$ ($\alpha/\beta = 1$). Also, starting from the macroscopic equation for the roughness the

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dynamical exponent has been theoretically calculated. They found $\beta = 0.629$ for the critical value $q_c = 0.539$.

In this paper, we use the TL model in order to investigate the imbibition of a viscous fluid in a porous media driven by capillary forces. We write a microscopic equation (ME), starting from the microscopic rules, for the evolution of the fluid height as a function of time. The ME allows us to identify two contributions that dominate the dynamics of the system, the 'diffusion' and the 'substratum' contributions. In this context we study the mean height speed (MHS), the interface activity density (IAD), i.e. the density of active sites of the interface, and the roughness as a function of time. We show that the diffusion contribution smooths out the surface for q well below the criticality but enhances the roughness near the critical value. To our knowledge, the separation into two contributions for all the quantities studied in this paper and the important role of the diffusion contribution to the critical power-law behaviour has never been studied before.

This paper is organized as follows. In section 2 we derive the microscopic equation for the evolution of height for the TL model. In section 3 we separate two contributions of the MHS: the diffusion and the substratum one. We find a relation between these contributions that allows us to write an analytical equation for the IAD. In section 4 the temporal derivative of square interface width as a function of time is derived from the ME and the two contributions are identified. These two contributions allow us to explain the mechanism of roughness. Finally, we conclude with a discussion in section 5.

2. The microscopic model

In the model introduced by TL [2] the interface growth takes place in a square lattice of edge L with periodic boundary conditions. We assign a random pinning force $g(\mathbf{r})$ uniformly distributed in the interval [0, 1] to every cell of the square lattice. For a given applied pressure $p > 0$, we can divide the cells into two groups: those with $g(\mathbf{r}) \leqslant p$ (free or active cells), and those with $g(\mathbf{r}) > p$ (blocked or inactive cells). Denoting by q, the density of inactive cells on the lattice, we have $q = 1 - p$ for $0 < p < 1$ and $q = 0$ for $p \geqslant 1$. The interface is specified completely by a set of integer column heights h_i ($i = 1, \ldots, L$). At $t = 0$ all columns are assumed to have the same height, equal to zero. During growth, a column is selected at random, say column i, and its height compared with those of its neighbouring columns $(i - 1)$ and $(i + 1)$. The growth event is defined as follows. If h_i is greater than either h_{i-1} or h_{i+1} by two or more units, the height of the lower of the two columns $(i - 1)$ and $(i + 1)$ is incremented by one (in the case of the two being equal, one of the two is chosen with equal probability). In the opposite case, $h_i < \min(h_{i-1}, h_{i+1}) + 2$, the column i advances by one unit provided that the cell to be occupied is an active cell. Otherwise no growth takes place. In this model, the time unit is defined as one growth attempt. In numerical simulations at each growth attempt the time t is increased by δt, where $\delta t = 1/L$. Thus, after L growth attempts the time is increased by one unit. In our simulations we used $L = 8192$ and a time interval much less than the crossover time to the static regime.

We consider the evolution of the height of the ith site for the process described above. We assume periodic boundary conditions in a one-dimensional lattice of L sites. At the time t a site j is chosen at random with probability $1/L$. Let us denote by $h_i(t)$ the height of the ith generic site at time t. The set of $\{h_i, i = 1, \ldots, L\}$ defines the interface between wet and dry cells. The time evolution for the interface in a time step $\delta t = 1/L$ is

$$h_i(t + \delta t) = h_i(t) + \frac{1}{L} G_i(h_{i-1}, h_i, h_{i+1}) \tag{1}$$

where

$$G_i = W_{i+1} + W_{i-1} + F_i(h_i')\, W_i \tag{2}$$

with

$$W_{i\pm1} = \Theta(h_{i\pm1} - h_i - 2)\{[1 - \Theta(h_i - h_{i\pm2})] + \delta_{h_i,h_{i\pm2}}/2\}$$
$$W_i = 1 - \Theta(h_i - \min(h_{i-1}, h_{i+1}) - 2).$$

Here $h_i' = h_i + 1$ and $\Theta(x)$ is the unit step function defined as $\Theta(x) = 1$ for $x \geqslant 0$ and equal to zero otherwise. $F_i(h_i')$ equals one if the cell at the height h_i' is active (i.e. the growth may occur at the next step) or zero if the cell is inactive. F_i is the interface activity function. G_i takes into account all the possible ways the site i can grow. The height in the site i is increased by one with probability:

(1) 1 if $j = i + 1$ and $h_{i+1} \geqslant h_i + 2$ and $h_i < h_{i+2}$

(2) $\frac{1}{2}$ if $j = i + 1$ and $h_{i+1} \geqslant h_i + 2$ and $h_i = h_{i+2}$

(3) 1 if $j = i - 1$ and $h_{i-1} \geqslant h_i + 2$ and $h_i < h_{i-2}$

(4) $\frac{1}{2}$ if $j = i - 1$ and $h_{i-1} \geqslant h_i + 2$ and $h_i = h_{i-2}$

(5) 1 if $j = i$ and $h_i < \min(h_{i-1}, h_{i+1}) + 2$ and $F_i(h_i') = 1$.

Otherwise, the height is not increased. The cases (1)–(4) are related to growth due to the neighbours of the site i. We shall call these mechanisms, growth by 'diffusion'. Note that these growths are not related to the disorder of the substratum. The factor $\frac{1}{2}$ takes into account the equality of first-neighbour heights at the $(i \pm 1)$th site in the cases (2) and (4). Case (5) is related to local growth, i.e., if the site i is chosen and the difference of heights between the ith and the lowest of its neighbours is less than two, then the height of the chosen site increases by one provided that the cell above the interface is active. We shall call this mechanism, growth by 'substratum'.

3. Mean height speed and interface activity density

Replacing $L = 1/\delta t$ and taking the limit $\delta t \to 0$, equation (1) becomes $\mathrm{d}h_i/\mathrm{d}t = G_i$. Averaging over the lattice we obtain ($h = \langle h_i \rangle$)

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \langle 1 - W_i \rangle + \langle F_i W_i \rangle. \tag{3}$$

This equation allow us to identify two separate contributions: diffusion, $\langle 1 - W_i \rangle$, and substratum, $\langle F_i\, W_i \rangle$†. Yang and Hu [6] defined two kinds of growth events: an event in which the growth occurs at the chosen site (type A—defined by us as substratum growth) and an event in which the growth occurs at the adjoint site (type B—our growth by diffusion). They counted, in numerical simulation, the event numbers, $N_A(t)$ of type A and $N_B(t)$ of type B, in a time interval L. They did not identify these terms as contributions to the mean height speed (MHS). Notice that $N_A(t) \propto \langle F_i\, W_i \rangle$ and $N_B(t) \propto \langle 1 - W_i \rangle$ (see figure 1). We shall see in section 4 that the separation of those two terms allows us to show how the diffusion enhances the roughness near the critical value. The separation into two contributions for all the quantities studied in this paper has never been done before.

The substratum contribution can be expressed as $f - \langle F_i\, (1 - W_i) \rangle$, where $f = \langle F_i \rangle$ is the IAD. We found an amazing numerical result:

$$\langle F_i(1 - W_i) \rangle = p \langle 1 - W_i \rangle. \tag{4}$$

† To compute equation (3) or any equation derived from the ME we froze the simulation at a time t. For this configuration we compute for the ith site all the contributions to the growth of this site in the next time $t + \delta t$ without changing the configuration. Then, we average over the lattice and over realizations. This technique has been employed in other systems, see [5].

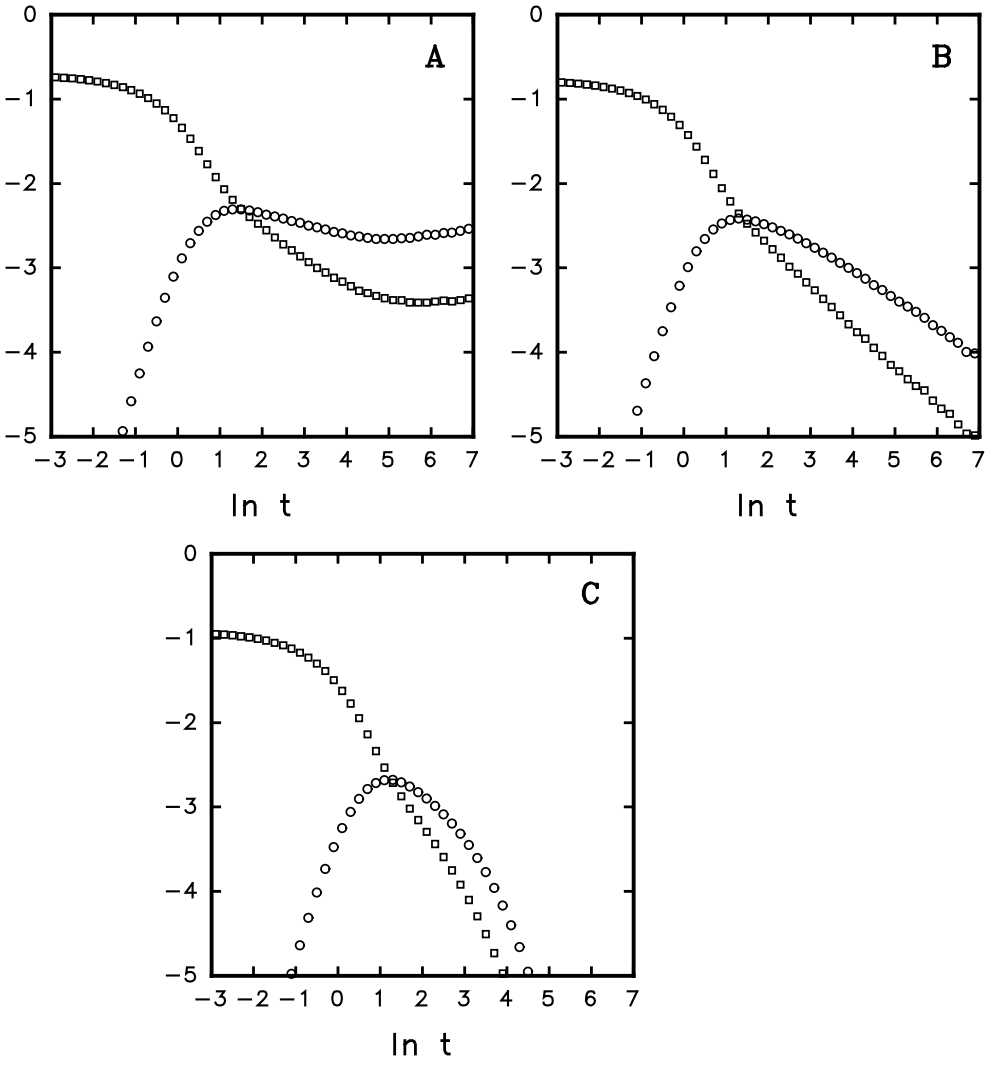

Figure 1. ln–ln plot of $\langle F_i W_i \rangle$ (□) and $\langle 1 - W_i \rangle$ (○) versus t. The parameter q is (a) 0.51 (b) 0.539 (c) 0.6.

We could not analytically obtain this result. Notice that F_i and $1 - W_i$ are not independent, and that $f \neq p$ for $t > 0$, as we shall see below. Using equation (4), the IAD is

$$f = p\langle 1 - W_i \rangle + \langle F_i W_i \rangle. \tag{5}$$

Figure 2 shows both sides of this equation as a function of time showing that equation (4) holds. Notice the similarity between equation (3) and (5). Figure 1 shows the diffusion and the substratum contributions as a function of the time for various values of q. At the initial time $\mathrm{d}h/\mathrm{d}t = f = p$. In the early time regime the substratum contribution dominates the diffusion one, because $1 - W_i$ is very small. The substratum contribution dominates the behaviour of f and $\mathrm{d}h/\mathrm{d}t$ in the early regime. As growth continues, the probability that growth will occur by diffusion becomes larger; the diffusion contribution increases and the substratum one decreases. This can be explained heuristically: inactive sites generate a difference of heights

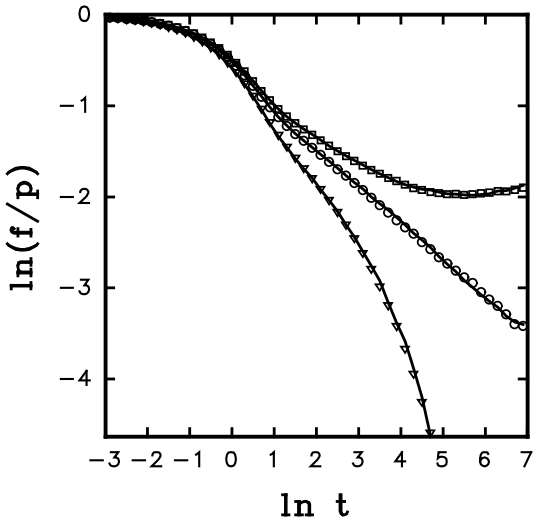

Figure 2. ln–ln plot of f/p versus t. The symbols show the right-hand side of equation (4) (in units of p), computed as explained in [5]. The full curve show the left-hand side of the same equation (in units of p) where $f = \langle F_i \rangle$. The parameter q is 0.51 (□), 0.539 (○) and 0.6 (▽). The critical case shows that the IAD goes as $t^{-\eta}$ with $\eta \simeq 0.40$.

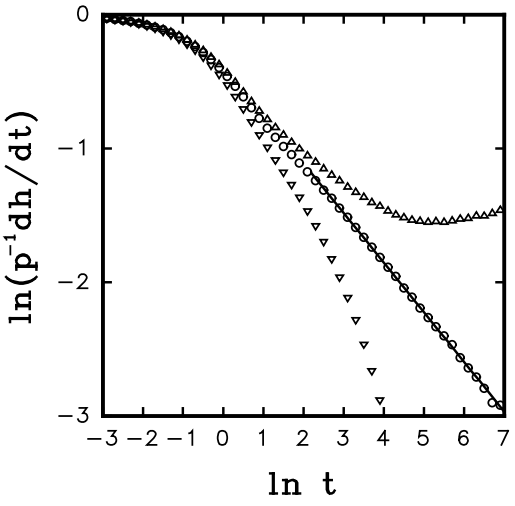

Figure 3. ln–ln plot of $p^{-1}\mathrm{d}h/\mathrm{d}t$ versus t. The parameter q is 0.51 (△), 0.539 (○) and 0.6 (▽). All cases shows the same behaviour in the early time regime. The subcritical case shows that the MHS asymptotically goes to certain constant. The critical case shows that the mean height goes as $t^{-\beta}$. The supercritical case shows that the mean height is asymptotically constant.

greater than two between any site and its neighbour, enhancing the growth by diffusion. As time goes on, long chains of pinned sites are generated, slowing down the diffusion contribution and hence the substratum one. For $q < q_c$ these contributions, which in turn dominate, saturate to equilibrium in the asymptotic regime; while, for $q \geqslant q_c$, both contributions go to zero because the system becomes pinned. At the critical value both contributions gives rise to a power law in the IAD and the MHS. Notice that only at the critical value does a power-law scaling hold for the MHS (see figure 3), which contradicts [2]. This was shown for the roughness by Braunstein and Buceta [4].

4. Roughness

From equation (1), the temporal derivative of the square interface width (DSIW) is:

$$\frac{\mathrm{d}w^2}{\mathrm{d}t} = 2\,\langle (h_i - \langle h_i \rangle) G_i \rangle. \tag{6}$$

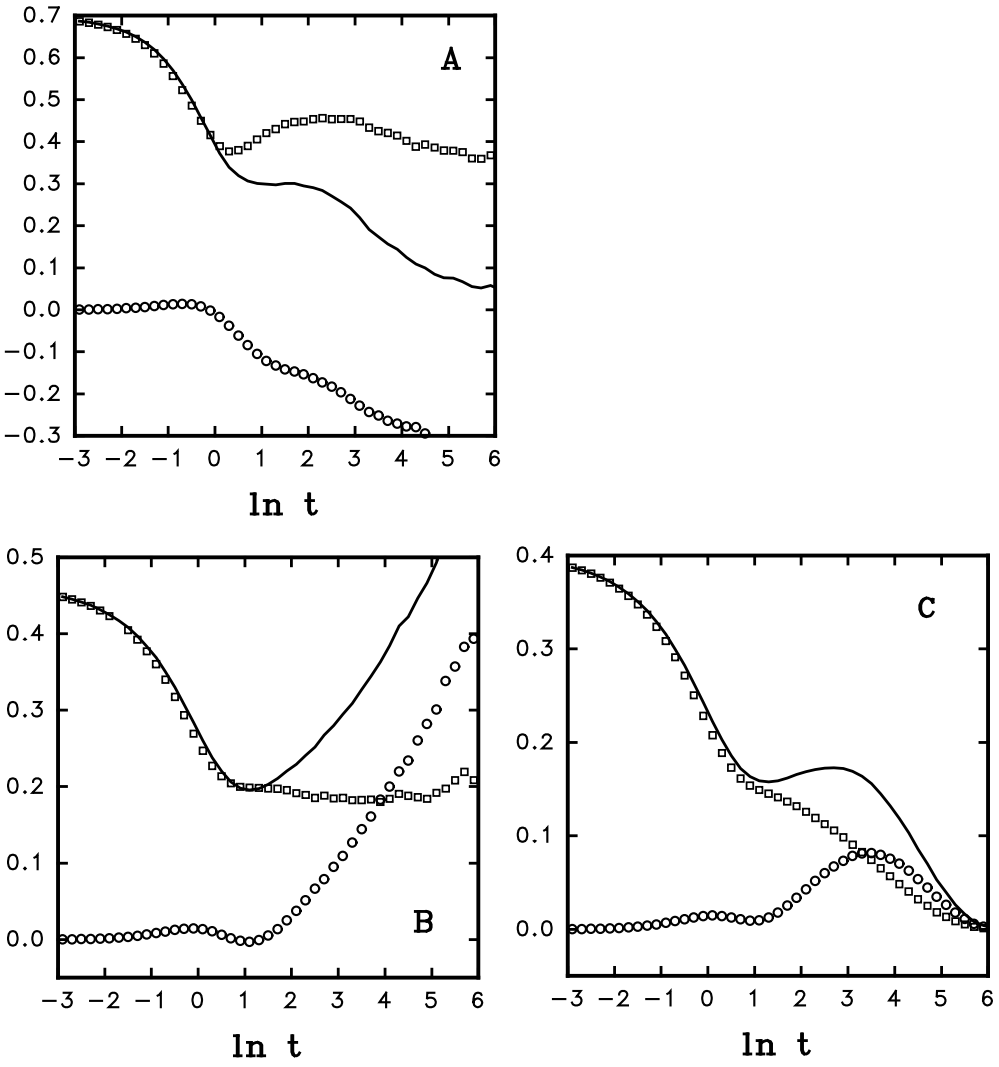

Figure 4. DSIW (full curve), and its diffusion (○) and substratum (□) contributions versus $\ln t$, for q equal to 0.3 (*a*), 0.539 (*b*) and 0.6 (*c*).

Replacing G_i from equation (2), the DSIW can also be expressed by means of substratum and diffusion additive contributions. The diffusion contribution is

$$2[\langle(1 - W_i)\min(h_{i-1}, h_{i+1})\rangle - \langle 1 - W_i\rangle\langle h_i\rangle] \tag{7}$$

and the substratum contribution is

$$2[\langle h_i F_i W_i\rangle - \langle h_i\rangle\langle F_i W_i\rangle] \tag{8}$$

where the relation $\Theta(x - x') + \Theta(x' - x) - \delta(x - x') = 1$ has been used to derive the diffusion contribution. In figure 4 we plot both contributions as a function of time for various values of q. At short times, the diffusion process is unimportant because Δh is mostly less than one. As t increases, the behaviour of this contribution depends on q. Notice, from equation (7), that the diffusion contribution may be either negative or positive. The negative contribution tends to smooth out the surface. Figure 4 shows that this case dominates for small q. The

positive diffusion contribution enhances the roughness. This last effect is very important at the critical value. At this value, the substratum contribution is practically constant, but the diffusion contribution is very strong, enhancing the roughness. This last contribution has important effects on the power-law behaviour. We think that it is amazing how the diffusion plays a dominant role in roughening the surface. To our knowledge, the strong effect on the roughness, at the criticallity, of the diffusion contribution has never been proven before.

Generally speaking, the substratum roughens the interface while the diffusion flattens it for small q, but the diffusion also roughens the interface when q increases. The diffusion is enhanced by substratum growth. The growth by diffusion may also increase the probability of substratum growth. This crossing interaction mechanism makes the growth by diffusion dominant near the criticality.

5. Conclusions

We wrote the ME for the evolution of the height in the TL model. The ME allows us to separate the substratum and the diffusion contributions and to explain the great interplay between them. We found that both contributions to the MHS are related in simple way. We found an amazing numerical result that allows us to derive the IAD in a simple way. The analytical proof of this numerical result is still open. All the quantities studied show the strong interplay of the diffusion and the substratum contribution in the dynamics. The substratum growth enhances the diffusion; increasing the growth by diffusion may increase the probability of substratum growth, and vice versa. This crossing interaction mechanism makes the diffusion contribution dominant at the criticality. The diffusion contribution of the DSIW shows different behaviour depending q. In the intermediate regime, when q is small, this contribution is negative, smoothing out the surface. It is astonishing that as q increases the contribution becomes positive, roughing the surface. Finally, we are sure that other DPD growth models would permit separation into two contributions with the same features of the TL model.

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